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Ordinal utility and the traditional theory of consumer demand[1]

Donald W. Katzner [University of Massachusetts/Amherst, USA]

In an earlier issue of this Journal, Jonathan Barzilai, in a paper entitled, "Inapplicable Operations on Ordinal, Cardinal, and Expected Utility" [see reference 2], has raised important issues regarding ordinal utility, and correctly clarified the meaning of the general notion of ordinality in terms of the mathematical theory of measurement. In that process, he has also subjected the traditional theory of consumer demand to serious attack. Barzilai's assault on traditional consumer theory, which is based on the mathematical theory of measurement, is useful because it brings to the fore the fact that, for economists, there is a second notion of ordinal utility, older than and independent of the mathematical-theory-of-measurement concept, and which is the relevant one for the traditional theory of consumer demand. That older approach seems to have had widespread acceptance among economists before the newer mathematical approach was known to them. The essence of Barzilai's attack consists of the claims that:

1. The function values of the utility function in that theory are ordinal and cannot, according to the mathematical theory of measurement, be subjected to arithmetic operations. This means that, since the reckoning of derivatives requires subtraction and division of function values, the derivatives of the ordinal utility function cannot be calculated. Marginal utilities, then, cannot exist, and hence the Lagrange-multiplier derivation of demand functions and their properties from constrained utility maximization is logically flawed and erroneous.
2. Both Hicks in *Value and Capital* [4] and Samuelson in his *Foundations of Economic Analysis* [11] base their discussions of the theory of consumer demand on differentiable, ordinal utility functions and the method of Lagrange multipliers. Their arguments too are, as a consequence of (1), logically flawed and erroneous. Since subsequent development of the theory has taken the same tack, the traditional theory of consumer demand as it was then and as it stands today is invalid.

The purpose of this paper is to demonstrate that these claims are based on a misunderstanding of the theory of consumer demand and the work of Hicks and Samuelson, and that the conclusion that that theory is logically flawed and invalid is unjustified. The misunderstanding arises in that, contrary to what is ordinarily done, Barzilai wants to use the theory-of-measurement notion of ordinal utility as the basis for the traditional theory of consumer demand.

I. Ordinal utility

One of the issues raised by Barzilai has to do with the meaning of the phrase "ordinal utility." Barzilai's approach is taken directly from the mathematical theory of measurement which provides, among other things, the technical requirements for constructing measures (e.g., Pfanzagl [9]). To describe what is involved in measurement from this perspective, consider

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ordinal in contrast with cardinal measurment. Let *A* be a set of objects, say, sticks of chalk. Suppose the relation longness orders the elements of *A* in that some sticks are seen to have more longness than others. Suppose also that one is able to "operate" on them in the sense of "combining longnesses" by lining up any two sticks end to end. An ordinal measure transforms longness into units of length. The ordinality guarantees that sticks with greater longnesses are assigned greater lengths when measured. A cardinal measure ensures additionally that when two sticks are lined up end to end, the length of the combined stick is the sum of the lengths of each separately.

From the perspective of the mathematical theory of measurement, the question of whether it is possible to measure utility ordinally or cardinally has to do with the kind of scale upon which the elements of the function values of the utility function are measured. In that context, let *A* be a collection of objects, say vectors or baskets of commodities, capable of providing what, for lack of a better term, may be called "pleasure" to the consumer. The set *A* is merely a general assemblage to be used as the basis for constructing a scale on which pleasure is measured in units of, say, utils. Assume that some ordering relation orders the objects of *A* by pleasure. Under certain technical restrictions on that ordering,[2] there exists an ordinal scale for measuring pleasure. When a utility function is present and when its function values are taken to be ordinally measured in this sense, the functions maps baskets of commodities into measured pleasure recorded as quantities of utils.

Now suppose that a combining operation is also defined on *A* that characterizes the means by which the pleasure in any two objects is to be consolidated into the pleasure of the combination. Then under additional, technical conditions,[3] the existence of a cardinal scale is assured and the pleasure obtained from any combination of two objects is measured in utils as the (possibly weighted) sum of the amounts of pleasure afforded by each separately.

Thus Barzilai's notion of ordinal utility requires a conceptual framework in which there is a direct connection to an underlying ordering of objects by pleasure. His definition of "ordinal space," over which the function values of his ordinal utility function range, is a follows: "An ordinal space *A* is a set of objects equipped only with the relations of order and equality" [2, p. 99]. As suggested above, the objects of *A* can be vectors or baskets of commodities. The order and equality relations he is referring to may be either the order and equality relations in the space *A* based on pleasure, or the greater-than and equal-to relations among the real-number quantities of utils in the space, *U*, of utility-function values. The two spaces are related by a function, the ordinal utility function, mapping the former into the latter that impresses on *U* the pleasure structure, and only that structure, of *A*. For him, the ranges of all ordinal utility functions are ordinal spaces.

Faithfully adhering to this notion of ordinality, Barzilai cannot permit arithmetic operations to be employed among quantities of utils. At first, this seems rather strange because quantities of utils are expressed as numbers and numbers can always be added, subtracted, multiplied, and divided. But what he most likely means here is that, although arithmetic operations can be applied to these numbers, they have no significance in terms of the underlying structure of pleasure described above. This is because, since utility values are only ordinal and not cardinal, there is no operation of combination in that underlying structure that will give meaning to the adding together (or the subtracting) of the quantities of utils of two objects. In

[2] See Pfanzagl [9, p. 75].
[3] *Ibid*., pp. 97-98.

[1, p. 61], Barzilai refers to what he sees as the logical necessity of having arithmetic operations, when they are to be employed, meaningfully represented in terms of the underlying pleasure structure as the “Principle of Reflection”.

In any case, it is on the basis of the mathematical-theory-of-measurement approach to ordinal utility that Barzilai concludes that the theory of consumer demand, including the arguments of Hicks and Samuelson, are flawed. The flaw arises in that, because the utility function is ordinal, the arithmetic operations needed to define the utility-function derivatives necessary in characterizing the relevant first-order constrained-utility maximization conditions cannot be supported by the Principle of Reflection. However, Barzilai's approach to ordinal utility, which is correct if one strictly adheres to the general notion of ordinality derived from the mathematical theory of measurement, is not the approach to ordinal utility taken by the traditional theory of consumer demand or by Hicks and Samuelson in their presentations of it. In these latter contexts, pleasure plays no role in relation to utility values. Indeed, utility values are not measures, in the theory-of-measurement sense, of anything. Rather the ordinal utility function is simply a numerical, differentiable[4] representation of a preference ordering (that includes the possibility of indifference and) that has no relation to any underlying pleasure structure. The starting point of the traditional theory of consumer demand is this preference ordering. That ordering is, perhaps, based on a judgment that, for some unspecified reasons, various baskets are better or no worse than others. Such a judgment need have nothing to do with pleasure and certainly has no connection to any underlying pleasure structure of the sort required by Barzilai.

The only meaning attributed to the ordinal utility representation of the preference ordering in the traditional theory of consumer demand is that if basket of commodities *b* is preferred to basket of commodities *c*, then the utility value assigned to *b* is greater than that assigned to *c*; and if the two baskets are indifferent, they are assigned the same utility value. The utility function values have no intrinsic meaning other than the information they provide concerning preferences. That is, the utility function contains exactly the same information as, and is an equivalent way of expressing the preference ordering. There is no underlying pleasure structure, and no necessity to have that structure and its properties, and only that structure and those properties, reflected in the meaning of, and in operations on, the utility function’s values. The fact that this function is often referred to as ordinal is not a reference to the notion of ordinality invoked by the mathematical theory of measurement and employed by Barzilai.[5] Of course, such a utility representation is not unique. Any increasing transformation of it (e.g., cubing) provides another utility function that represents the same preference ordering in the same sense as that of the original representation. And this implies that marginal utilities, although calculable, have no meaning with respect to the specified preference ordering. Applying an increasing transformation to the utility function changes the marginal utilities too. (Since the utility function has no relation to an underlying pleasure structure, neither do the marginal utilities.)

[4] Actually, twice, continuous differentiability is often assumed.
[5] In some of my earlier work (e.g., Katzner [6, pp. 49-50]), the distinction between these two approaches to ordinal utility and the fact that economists subscribe to the one that eschews any reference to an underlying pleasure structure is obscured. Barzilai's paper has led me to clarify the matter here.

Indifference surfaces are defined in terms of the indifference relation that is included in the preference ordering.[6] A utility function is not required to characterize them. Assuming differentiability, their "slopes" (partial derivatives), referred to as (the negatives of) marginal rates of substitution, are the rates at which the consumer can substitute at the margin one good for another and remain on the same indifference curve. Even though those slopes can be expressed in terms of the ratio of meaningless marginal utilities, they depend only on the indifference relation – not the specific utility function employed. Increasing transformations of the utility function change the marginal utilities, but not their ratio. Meaningful marginal utilities are not needed to characterize marginal rates of substitution.

The arguments of Hicks and Samuelson take this perspective on ordinal utility. In the mathematical appendix of *Value and Capital* [4, pp. 305-306], Hicks first derives the first-order conditions for constrained maximization of the utility function using the method of Lagrange. Then, when commenting on the "ordinal character of utility" in the section of the mathematical appendix with that title, Hicks asserts that, "The equilibrium conditions [first-order maximization conditions] … for the consumer … do not depend upon the existence of any unique utility function" and the fact that they appear in terms of marginal utilities is only "… the most convenient way to write them" [4, p. 306]. In the text itself he states: "The quantitative concept of utility is not necessary in order to explain market phenomena," and in reference to the theory of consumer demand: "We start off from the indifference map alone; nothing more can be allowed" [4, p. 18]. Hicks then goes on in the text to show how the theory of consumer demand can be set out in non-mathematical terms without any reference to a utility function. On p. 19, Hicks acknowledges that although the notion of marginal utility has no meaning, the ratio of marginal utilities does. The word "ordinal" does not appear on these pages. The facts that the word "ordinal" is used only once (as quoted above) in the mathematical appendix as part of the title of a section, and that it appears in the index in reference to pp. 17 and 18 without actually appearing in the text on pp. 17 and 18, suggests that Hicks used it only as an afterthought, perhaps to bring his work more in line with the latest terminology in vogue at the time. There is nothing to suggest an underlying pleasure structure and a concept of ordinal utility similar to that of the theory of measurement invoked by Barzilai. Indeed, if there were, then his attack on Hicks would be fully justified.

Samuelson [11] uses the phrase "ordinal preference field" on p. 93. He explains what he means by this on p. 94: "For any two combinations of goods [baskets of commodities or vectors *x* and *y*] … it is only necessary that the consumer be able to place them in one of the following categories: (a) x preferred to y, (b) y preferred to x, [or] (c) x and y equally preferred or indifferent. For convenience, we may attach a number to each combination; this is assumed to be a continuous differentiable function."[7] In relating utility values to the preference ordering, he then gives the exact same statement of the non-mathematical-theory-of-

[6] Let the space of commodities *X* be the collection of all nonnegative baskets of goods. A preference ordering $\geq_x$ is a binary relation defined on *X* that is reflexive and transitive. (Definitions of the latter two terms appear in n. 8 below.) The (strict) preference relation $>_x$ and the indifference relation $=_x$ are separated from the combined preference-indifference relation $\geq_x$ as follows: In the case of (strict) preference, for all x' and x'' in *X*, $x' >_x x''$ if and only if $x' \geq_x x''$ and it is not the case that $x'' \geq_x x'$. And, with regard to indifference, for all x' and x'' in *X*, $x' =_x x''$ if and only if both $x' \geq_x x''$ and $x'' \geq_x x'$.

[7] I have taken a slight liberty here in changing Samuelson's mathematical notation.

measurement meaning of the concept of ordinal utility representation as provided here. Once again, there is nothing to suggest anything that would indicate a concept of ordinality similar to Barzilai's. Samuelson's approach to the theory of demand is similar to that of Hicks presented in the mathematical appendix of *Value and Capital*.

There is further evidence in the literature suggesting that Hicks and Samuelson were not confused about the notion of ordinality they employed, and were thinking of that concept only as a representation of a preference ordering – not in terms of the mathematical theory of measurement conceptualization as claimed by Barzilai. In a 1985 volume called *Abstract Measurement Theory*, Narens [8] traces the history of Barzilai's mathematical-theory-of-measurement approach to measurement. On p. 5 Narens says, "The view that measurement consists in specifying homomorphisms of some qualitative (or empirical) structure into a numerical one [*i.e.*, the mathematical-theory-of-measurement or Barzilai approach] is called the representational theory of measurement, and since the late 1950s it has gained widespread support among measurement theorists." Indicating that the representational theory of measurement found its way outside of the measurement theory world also in the 1950s, Luce and Narens write [7, p. 220], "More than anyone else, Suppes brought to the attention of non-mathematicians this axiomatic style of studying the measurement of attributes." And the papers of Suppes they cite in this regard appeared starting in the 1950s. Before the 1950s, the common notion of ordinal scales seems to have been that described by Stevens [12, p. 679] in 1946: "The ordinal scale arises from the operations of rank ordering… any 'order-preserving' transformation will leave the scale form invariant ." He goes on to say that, "In the strictest propriety the ordinary statistics involving means and standard deviations ought not to be used with these scales, for these statistics imply a knowledge of something more than the relative rank-order of data. On the other hand, for this 'illegal' statisticising there can be invoked a kind of pragmatic sanction: In numerous instances it leads to fruitful results."

Given today's knowledge of the matter, one may question that such results are, in fact, fruitful. But, in any case, Stevens is not saying that arithmetic operations are not legitimate with ordinal numbers; only that the results obtained when calculating means and standard deviations of ordinal data, which require the use of arithmetic operations, should be used with care. Care should be taken because applying an increasing transformation to the data will not change the meaning of the data but may change the results of the calculation.

Now Hicks was writing about the theory of consumer demand in the 1930s and Samuelson in the 1930s and early 1940s. (Samuelson's *Foundations* was completed in 1941 but, due to World War II, not published until 1947.) This suggests that the illegitimacy of applying arithmetic operations to ordinal numbers in the mathematical-theory-of-measurement context was not known to them, and that their attitudes toward ordinal numbers were likely to be similar to the non-algebraic approach of Stevens rather than the homomorphism approach of Barzilai. That is, as with many other scholars both in and outside of economics at the time, it would probably not have occurred to them that the application of arithmetic operations to ordinal numbers might be improper since their vision of ordinality had only to do with the fact that applying increasing transformations to ordinal numbers yields new numbers having the same informational content as the old. From the Hicks-Samuelson perspective then, there would be nothing wrong in using the term 'ordinal' in conjunction with their order-preserving utility function. After all, order-preserving functions arise from rank orderings and order-preserving transformations have no impact on the underlying ordering – properties similar to those described for ordinal numbers in the first quotation cited from Stevens above. Thus it seems that Hicks and Samuelson were not using ordinality in the sense of Barzilai and were thinking of the utility function simply as an order-preserving representation of a preference ordering.

II. Consumer demand functions

As indicated in the quotations and discussion attributed to Hicks above, the traditional theory of consumer demand can be stated in its entirely omitting reference to utility of any kind. Without becoming too deeply involved in technicalities, here is one mathematical account of it:

Begin with a preference ordering defined among commodity baskets (vectors) in the non-negative orthant of Euclidean space (the space of commodities) that is reflexive, transitive, total, increasing, and strictly convex.[8] Assume that through each basket of commodities in that space there is a continuous indifference surface defined in terms of the indifference relation (mathematically characterized in n. 6 above).

Now every vector of (strictly) positive prices and income determines a budget set, *B*, defining the collection of baskets available to the consumer for purchase. Budget sets are compact.[9] Consider one such vector of prices and income and hence a specific set *B*. For each basket x' in that set there is another set, C', of all baskets preferred or indifferent to x'. The intersection of the latter set (which is closed because indifference surfaces are assumed continuous) and the budget set, $B \cap C'$, is also compact. Two baskets x' and x'' on the same indifference surface yield the same intersection $B \cap C'$. The collection of all sets of the form $B \cap C'$, one set corresponding to each basket x' in the budget set, has the finite intersection property, namely that the intersection of every non-empty finite sub-collection of the collection of all sets of the form $B \cap C'$ is non-empty. It follows that[10] there exists a basket x^0 in the budget set that is contained in all sets of the form $B \cap C'$. That basket is unique in the budget set and (strictly) preferred to all other baskets in it. Loosely speaking, x^0 is the "most preferred" basket in the budget set *B*.

It turns out that under the assumptions on preferences stated above, a continuous (not necessarily differentiable) utility representation of the preference ordering exists. That utility functions is sometimes called ordinal. But it is not ordinal in the sense of the mathematical theory of measurement and of Barzilai. Rather, it is ordinal in the sense of being a representation of a preference ordering. In terms of such a utility function, the basket that is (strictly) preferred to all other baskets in the budget set can be said to maximize utility subject to the budget constraint. But, of course, that utility function and the constrained maximization

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[8] A relation such as $\geq_x$ is *reflexive* when $x \geq_x x$ for all *x* in the space of commodities *X*. It is *transitive* if $x' \geq_x x''$ and $x'' \geq_x x'''$ imply $x' \geq_x x'''$, for all x', x'', and x''' in *X*. And it is *total* provided that, for all x' and x'' in *X*, either $x' \geq_x x''$ or $x'' \geq_x x'$. The relation $\geq_x$ is *increasing* whenever $x' \neq x''$ and $x' \geq x''$ (in terms of inequality and greater-than-or-equal-to among Euclidean vectors) implies $x' >_x x''$ for all $x' > 0$ and $x'' > 0$, where $>_x$ is defined in n. 6 above. It is *strictly convex* if $\theta x' + (1-\theta)x'' >_x x'$ for all $x' > 0$ and $x'' > 0$ such that $x' =_x x''$ and $0 < \theta < 1$.

[9] A set is compact is if it is closed (*i.e.*, contains all of its limit points) and bounded.

[10] See, for example, Hall and Spenser [3, p. 68].

of it is irrelevant in, and imposes no additional restrictions on, the preceding derivation of the most preferred basket in each budget set *B*.

Consumer demand functions are now defined as that function that relates to each price-income vector, the basket of commodities that is (strictly) preferred to all other baskets in the budget set as determined above by the specified price-income vector. All of the standard properties of demand functions, such as homogeneity of degree zero and continuity, can now be proved. Even without a utility function or its differentiability if one is in view, demand functions may still be differentiable and, where they are, the well-known negative definiteness and symmetry properties follow.[11] Clearly, then, the theory of consumer demand does not need to rely on a utility function differentiable or otherwise for its logical viability.[12]

Of course, there is little doubt that the theory of consumer demand has its limitations and can be subjected to justifiable criticisms. But that it contains logical flaws of the sort described by Barzilai is not one of the latter.

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Author contact: dkatzner@econs.umass.edu

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[11] See, for example, Katzner [5, pp.110-112].
[12] An account of the theory of consumer demand similar to that presented here has been given by McKenzie [10].