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The giant blunder at the heart of General Equilibrium Theory

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Abstract
Proofs of general equilibrium crucially hinge on establishing the existence of an equilibrium price vector that makes excess demand in all markets equal to zero. This paper shows that the “price vector” is not a vector and that all proofs of general equilibrium are, therefore, invalid.

1. Introduction

The publication of Paul Samuelson’s *Foundations of Economic Analysis* in 1947 came as a shock to most economists of the day. Accustomed to thinking in terms of words and diagrams, à la Marshall, they had believed that mathematical economics was one of the ways of doing economics. But now they were told that it was the best way. The epigraph on the title page summed up the author’s views on using math in economics; it approvingly quoted physicist Josiah Willard Gibbs: “Mathematics is a language.” The book was based on Samuelson’s Ph.D. thesis, and the difficulty it presented to older economists may be judged from the anecdote which had Joseph Schumpeter, chairman of the thesis examining committee, asking the other members after Samuelson’s presentation, “Well, gentlemen, did we pass?”

By the end of the 1950s economists had begun to absorb and apply the ideas underlying *Foundations*. Indeed, the optimising methods in it had become conventional wisdom. It was then that economists were struck by another mathematical thunderbolt, this time in the form of modern General Equilibrium Theory (GET). The new theory drew on areas of mathematics that no one would have thought possible to connect to economics: sets, topology, fixed-point theorems, and so on. If the calculus of optimisation was difficult for the uninitiated, the math of GET was positively weird.

The new ideas began as a dribble of papers in the early 1950s, had widened into a stream by the end of the decade, and then became a flood. Soon they were being routinely taught to graduate students and became a part of microeconomic price theory. Most important of all, they were a showcase for the claim that the use of mathematics could resolve questions that had vexed economists for ages, economists who did not have access to the toolkit of modern mathematics. If Samuelson had shown that mathematical economics was the best way of doing economics, GET went further to underline that mathematical economics was the only way.

Over the past half century GET has accounted for at least two Nobel prizes in economics, hundreds if not thousands of published papers, and tens of thousands of academic lecture hours. As this paper will show, all this was an utter waste of time and resources because GET is founded on a basic error committed by the pioneers in the subject like Kenneth Arrow, Gerard Debreu and Lionel McKenzie. And those who came after them were merely parroting the errors they had been taught, and unthinkingly passing them on to future generations.

2. The price vector

The idea of a price vector is crucial to GET, as will be clear from the following quotes, taken from Gerard Debreu's Nobel Prize lecture (Debreu, 1983).

> "The most primitive of the concepts of the theory I will survey and discuss is that of the commodity space. One makes a list of all the commodities in the economy. Let *l* be their finite number. Having chosen a unit of measurement for each one of them, and a sign convention to distinguish inputs from outputs (for a consumer inputs are positive, outputs negative; for a producer inputs are negative, outputs positive), one can describe the action of an economic agent by a vector in the commodity space R^l. The fact that the commodity space has the structure of a real vector space is a basic reason for the success of the mathematization of economic theory. In particular convexity properties of sets in R^l, a recurring theme in the theory of general economic equilibrium, can be fully exploited. If, in addition, one chooses a unit of account, and if one specifies the price of each one of the *l* commodities, one defines a price-vector in R^l, a concept dual to that of a commodity-vector. The value of the commodity-vector **z** relative to the price-vector **p** is then the inner product **p.z**."

> "One of the aims of the mathematical theory that Walras founded in 1874-77 is to explain the price-vector and the actions of the various agents observed in an economy in terms of an equilibrium resulting from the interaction of those agents through markets for commodities. In such an equilibrium, every producer maximizes his profit relative to the price-vector in his production set; every consumer satisfies his preferences in his consumption set under the budget constraint defined by the value of his endowment-vector and his share of the profits of the producers; and for every commodity, total demand equals total supply. Walras and his successors for six decades perceived that his theory would be vacuous without an argument in support of the existence of at least one equilibrium, and noted that in his model the number of equations equals the number of unknowns, an argument that cannot convince a mathematician. One must, however, immediately add that the mathematical tools that later made the solution of the existence problem possible did not exist when Walras wrote one of the greatest classics, if not the greatest, of our science."

And a little later, Debreu describes how the idea of the price vector combines with elements of topology to yield a proof of the existence of an equilibrium price vector:

> "In the summer of 1950, Arrow, at the Second Berkeley Symposium on Mathematical Statistics and Probability, and I, at a meeting of the Econometric Society at Harvard, separately treated the same problem by means of the theory of convex sets. Two theorems are at the center of that area of welfare economics. The first asserts that if all the agents of an economy are in equilibrium relative to a given price-vector, the state of the economy is Pareto-optimal. Its proof is one of the simplest in mathematical economics. The second provides a deeper economic insight and rests on a property of convex sets. It asserts that associated with a Pareto-optimal state s of an economy, there is a price-vector **p** relative to which all the agents are in equilibrium (under conditions that, here as elsewhere, I cannot fully specify). Its proof is based on the observation that in the commodity space R^l, the a priori given endowment vector **e** of the economy is a boundary point of the set E of all the endowment vectors with which

> it is possible to satisfy the preferences of all consumers at least as well as in the state s. Under conditions insuring that the set E is convex there is a supporting hyperplane H for E through **e**. A vector **p** orthogonal to the hyperplane H, pointing towards E has all the required properties. The treatment of the problem thus given by means of convexity theory was rigorous, more general and simpler than the treatment by means of the differential calculus that had been traditional since Pareto."

If all this sounds incomprehensible it does not really matter. Our aim in reproducing these quotes is to underline the importance that the price vector plays in GET and, also, why Arrow, Debreu and others fell into the error of assuming that an n-tuple of prices is a vector.

3. A little about vectors

A vector in physics is defined as a quantity having both magnitude and direction. It is represented, as in the figure below, as a line segment with an arrowhead at one end.

Figure 1: A vector has both magnitude and direction

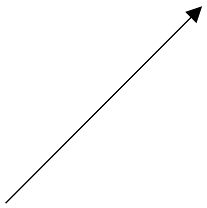
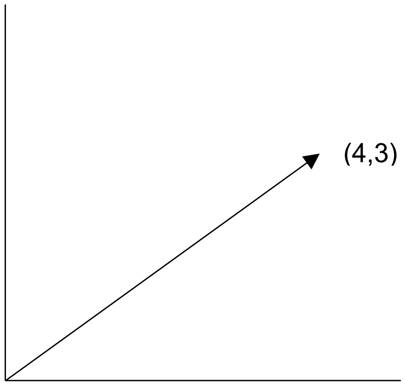


Figure 2: A vector in a Cartesian coordinate system



A vector can be placed in a Cartesian coordinate system as shown above.
The vector represents a force acting on a body. It has magnitude 5 newtons and its components along the x-axis and y-axis have magnitude 4 newtons and 3 newtons respectively. In defining the vector by the coordinates of one of its end points, the other being at the origin, we specify both the magnitude and the direction of the vector.

The convenience is immediately obvious. If one vector is defined as (4, 3) and another is defined as (3, 4), then the vector formed by the addition of the two vectors is (7, 7). Its magnitude is $\sqrt{98}$ and it forms an angle of 45º with the x-axis.

The idea can be extended to three dimensions. If three vectors defined as (2, 3, 4), (5, 6, 7) and (8, 9, 10) are added, then the combined vector will have coordinates (15, 18, 21).

Vectors can also be multiplied with one another. A particularly useful multiplication is what is called a dot product or an inner product. Thus, if a body is acted on by a force defined as (f_x, f_y, f_z) causing a displacement defined by (s_x, s_y, s_z), then the work done by the force is $f_xs_x + f_ys_y + f_zs_z$.

The 3-tuples representing quantities in 3-dimensional Euclidean space can be extended to n-dimensions. Thus, one n-tuple Q could represent the quantities of *n* goods (q_1, q_2, q_3, … , q_n) consumed in an economy and another n-tuple P could represent the prices of those goods (p_1, p_2, p_3, … , p_n). The total value of the goods consumed in terms of money would then be $q_1p_1 + q_2p_2 + q_3p_3 + \ldots + q_np_n$. There is a clear parallel with the scalar representing work, being the inner product of the 3-tuple representing force and the 3-tuple representing displacement.

Given this parallel it is not surprising that the pioneers of GET jumped to the conclusion that the n-tuple representing quantities of goods and the n-tuple representing the prices of those goods are both vectors. To show why that is erroneous we must take a short digression through the idea of temperature.

4. A meditation on temperature

Sometime in the 1960s, Richard Feynman, the physicist, was asked to serve on a committee to help select the books that would be used in schools in California. He was sent a huge pile of books for which he set up a special bookshelf. The books were lousy, and as Feynman read them in the basement his wife would hear sounds from below like that of a volcano exploding (Feynman, 1985).

> “Finally I come to a book that says, ‘Mathematics is used in science in many ways. We will give you an example from astronomy, which is the science of stars.’ I turn the page, and it says, ‘Red stars have a temperature of four thousand degrees, yellow stars have a temperature of five thousand degrees. . .’ — so far, so good. It continues: ‘Green stars have a temperature of seven thousand degrees, blue stars have a temperature of ten thousand degrees, and violet stars have a temperature of ... (some big number).’ There are no green or violet stars, but the figures for the others are roughly correct. It's vaguely right — but already, trouble! …"
>
> “Anyway, I'm happy with this book, because it’s the first example of applying arithmetic to science. I'm a bit unhappy when I read about the stars' temperatures, but I'm not very unhappy because it's more or less right — it's just an example of error. Then comes the list of problems. It says, ‘John and his father go out to look at the stars. John sees two blue stars and a red star. His father sees a green star, a violet star, and two yellow stars. What is the total temperature of the stars seen by John and his father?’ — and I would explode in horror.”
>
> “My wife would talk about the volcano downstairs. That's only an example: it was perpetually like that. Perpetual absurdity! There's no purpose whatsoever in adding the temperature of two stars. Nobody ever does that except, maybe, to then take the average temperature of the stars, but not to find out the total temperature of all the stars! It was awful! All it was was a game to get you to add, and they didn't understand what they were talking about.”

Both the mass of an object and its temperature are measured by real numbers together with some unit. But whereas the masses of two objects can be added it makes no sense to add the temperatures of two objects. Even transferring an operation as simple as addition from mathematics to the real world requires you to first ask whether the operation is valid.

5. What are vectors?

In the last but one section, we defined a vector as a quantity having both magnitude and direction and represented it by an arrow. This makes sense in Euclidean 3-dimensional space. But in higher dimensions the idea of direction is not intuitive and we need a more formal definition that is consistent with the definition in three dimensions. In mathematics, an object is defined as a vector if it is an element in a vector space. This seems a circular definition but the additional requirements make it clear why it is defined in this way. Thus, when a vector is multiplied by a scalar (a real number, for our purpose) the result must be an element of the vector space, i.e., another vector. And a vector added to another vector must also be a vector in that vector space.

Consider two 10-tuples of numbers, $T_1 = (t_1, t_2, t_3, \dots, t_{10})$ and $T_1' = (t_1', t_2', t_3', \dots, t_{10}')$. Let these represent the temperatures at ten points along the lengths of two metal bars. Then it is obvious that these 10-tuples cannot be vectors because they fail the requirement of vector addition; it makes no sense to add t_1 and t_1' because, as shown in the section on temperature, adding temperatures is a meaningless operation.

Similarly, consider two 10-tuples of numbers $P_1 = (p_1, p_2, p_3, \dots, p_{10})$ and $P_1' = (p_1', p_2', p_3', \dots, p_{10}')$. Let P_1 be the prices of 10 goods which an individual consumes on Monday and P_1' be the prices of the same goods which he consumes on Tuesday. Now it is meaningless to add the price of a good on Monday to the price of the good on Tuesday. Therefore, it is meaningless to add the elements of the two 10-tuples P_1 and P_1'. Hence, the two 10-tuples cannot be vectors and, indeed, there cannot be such a thing as a price vector.

We are therefore forced to conclude that GET, which in its modern version is nearly three quarters of a century old, is merely highfalutin nonsense.

6. A bit of history

Einstein described Gibbs as “the greatest mind in American history”. And later, when asked who were the most powerful thinkers he had known, Einstein said: “Lorenz”, adding, “I never met Willard Gibbs; perhaps had I done so, I might have placed him beside Lorenz.”

Gibbs’s influence on modern economics, especially on optimisation theory, is well known. That influence was through the effect on Samuelson, via Gibbs’s pupil, E.B. Wilson.

But Gibbs, along with Oliver Heaviside, was also the inventor of vectors, the subject matter of this paper. So, his influence runs through the other important strand of modern mathematical economics as well, mainly in proofs of general equilibrium, though the price vector has since ramified into other areas such as international trade theory.

Critics of mathematical economics would say that Gibbs was doubly unfortunate in his disciples, at one remove. But, of course, that was hardly his fault.

Samuelson must be blamed for erecting a huge mathematical edifice, without first ascertaining that the utility and profit functions are differentiable.

In GET, the error that Arrow and Debreu made was in blindly transferring mathematical ideas to the real world without first ascertaining that those ideas were transferable. It is significant that both of them were mathematicians who wandered into economics.

7. Conclusion

Debreu noted in his Nobel Prize lecture that the success of the mathematization of economic theory depended "on the fact that the commodity space has the structure of a real vector space". We have shown that this is incorrect. The "price vector" is not a vector, and GET is therefore false. But we may go further and assert that not only was the proof incorrect, what was set out to be proved was not true in the first place. The real economy cannot be brought into equilibrium by adjusting prices. And indeed, the real economy is never in equilibrium.

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